

Matching criticality in intersecting hypergraphs*

Liyang Kang¹, Zhenyu Ni¹, Erfang Shan^{1,2†}

¹Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China

²School of Management, Shanghai University, Shanghai 200444, P.R. China

Abstract

The transversal number $\tau(H)$ of a hypergraph H is the minimum cardinality of a set of vertices that intersects all edges of H . The matching number $\alpha'(H)$ of H is the size of a largest matching in H , where a matching is a set of pairwise disjoint edges in H . A hypergraph is intersecting if each pair of edges has a nonempty intersection. Equivalently, H is an intersecting hypergraph if and only if $\alpha'(H) = 1$. We observe that $\tau(H) \leq r$ for an intersecting hypergraph H of rank r . For an intersecting hypergraph H of rank r without isolated vertex, we call H 1-special if $\tau(H) = r$; H is maximal 1-special if H is 1-special and adding any missing r -edge to H increases the matching number. Furthermore, H is called 1-edge-critical if for any $e \in E(H)$ and any $v \in e$, v -shrinking e increases the matching number; H is called 1-vertex-critical if for every vertex v in H deleting the vertex v and v -shrinking all edges incident with v increases the matching number. The intersecting hypergraphs, as defined above, are said to be matching critical in the sense that the matching number increases under above definitions of criticality [M.A. Henning and A. Yeo, Quaest. Math. 37 (2014) 127–138]. Let $n^i(r)$ ($i = 2, 3, 4, 5$) denote the maximum order of a hypergraph in each class of matching critical intersecting hypergraphs. In this paper we study the extremal behavior of matching critical intersecting hypergraphs. We show that $n^2(r) = n^3(r)$ and $n^4(r) = n^5(r)$ for all $r \geq 2$, which answers an open problem on matching critical intersecting hypergraphs posed by Henning and Yeo. We also give a strengthening of the result $n^4(r) = n^5(r)$ for intersecting r -uniform hypergraphs.

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[†]Corresponding authors. Email address: efshan@shu.edu.cn (E. Shan)

1 Introduction

The relationship between transversals and matchings in hypergraphs have been extensively studied in [4, 8, 11, 17, 18] and elsewhere. In this paper, we study transversals and matchings in intersecting hypergraphs.

Hypergraphs are a natural generalization of undirected graphs in which “edges” may consist of more than 2 vertices. More precisely, a (finite) *hypergraph* $H = (V, E)$ consists of a (finite) set V and a collection E of non-empty subsets of V . The elements of V are called *vertices* and the elements of E are called *hyperedges*, or simply *edges* of the hypergraph. If there is a risk of confusion we will denote the vertex set and the edge set of a hypergraph H explicitly by $V(H)$ and $E(H)$, respectively. An *r-edge* is an edge containing exactly r vertices. The *rank* of a hypergraph H is the maximum size of an edge in H . Specially, An *r-uniform* hypergraph H is a hypergraph such that all edges are r -edges. Obviously, every (simple) graph is a 2-uniform hypergraph. Throughout this article, all edges have size at least 2.

Two vertices u and v of H are *adjacent* in $H = (V, E)$ if there is an edge e in H such that $u, v \in e$. A vertex v and an edge e of H are *incident* if $v \in e$. The *degree* of a vertex $v \in V$, denoted by $d_H(v)$ or $d(v)$ for shortly if H is clear from the context, is the number of edges incident to v . The *minimum degree* among the vertices of H is denoted by $\delta(H)$. Two edges in H are said to be *overlapping* if they intersect in at least two vertices. The *quasidegree* of a vertex v of H , denoted by $qd_H(v)$ or simply by $qd(v)$, is the maximum number of edges in H whose pairwise intersection is only v . Two vertices u and v are *connected* if there exists a sequence $u = u_0, u_0, u_1, \dots, u_k = v$ of vertices of H in which u_{i-1} is adjacent to u_i for $i = 1, 2, \dots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected.

A set $T \subseteq V$ is called *transversal* (also called *vertex cover*) of $H = (V, E)$ if it intersects every edge of H , i.e., $T \cap e \neq \emptyset$ for all $e \in E$. The minimum cardinality of the transversals, denoted by $\tau(H)$, is called the *transversal number* (also called *covering number*) of H . A subset $M \subseteq E$ is a *matching* if every pair of edges from M has an empty intersection. The maximum cardinality of a matching M is called the *matching number*, denoted by $\alpha'(H)$. If M is a matching in H , then we call a vertex that belongs to an edge of M an *M-matched vertex*. Transversals and matchings in hypergraphs have been extensively studied in the literatures (see, for example, [2, 3, 6, 14, 15, 16, 20, 21, 23]).

A hypergraph H is *k-colorable* if there exists a coloring of the vertices of H using k colors such that there is no monochromatic edge. A hypergraph H is *k-chromatic* if t is the smallest value for which H is k -colorable.

For a subset $E' \subseteq E(H)$ of edges in H , we define $H - E'$ to be the hypergraph obtained from H by deleting the edges in E' and resulting isolated vertices, if any. If $E' = \{e\}$, then we write

$H - E'$ simply as $H - e$. If we remove the vertex v from the edge e , we say that the resulting edge is obtained by v -shrinking the edge e .

A hypergraph is called *intersecting* if any two edges have nonempty intersection. Clearly, H is intersecting if and only if $\alpha'(H) = 1$. Even for intersecting hypergraphs, a long-standing open problem known as Ryser's Conjecture is open for all $r \geq 6$. Intersecting hypergraphs are well studied in the literature (see, for example, [1, 5, 7, 10, 12, 19, 22]).

In [18] Henning and Yeo introduced five classes of matching critical intersecting hypergraphs. In this paper we restrict our attention to intersecting hypergraphs with the matching criticality. We study the extremal behavior of the matching critical intersecting hypergraphs.

It is clear that if H has rank r then $\tau(H) \leq r\alpha'(H)$, and this is attained for example by the complete r -uniform hypergraph K_{2r-1}^r with $2r - 1$ vertices, which has $\tau(K_{2r-1}^r) = r$ and $\alpha'(K_{2r-2}^r) = 1$. In particular, if H is an intersecting hypergraph with rank r , then $\tau(H) \leq r$. Motivated by this observation, Henning and Yeo [18] gave the following definition.

Definition 1.1. *For $r \geq 2$, an intersecting hypergraph H of rank r with $\delta(H) \geq 1$ is 1-special if $\tau(H) = r$. Further, H is maximal 1-special if H is 1-special and adding any missing r -edge to H increase the matching number.*

We remark that a 1-special intersecting hypergraph of rank r must be r -uniform, since every edge of H is a transversal of H .

The two other families of matching criticality in hypergraphs, namely α' -edge-criticality and α' -vertex-criticality, are defined in [18].

Definition 1.2. *For a hypergraph H , H is α' -edge-critical if for any $e \in E(H)$ and any $v \in e$, v -shrinking e increases the matching number. H is α' -vertex-critical if for every vertex v in H deleting the vertex v and v -shrinking all edges incident with v increases the matching number. In particular, when $\alpha'(H) = 1$, α' -edge-critical and α' -vertex-critical are simply called 1-edge-critical and 1-vertex-critical, respectively.*

By Definition 1.2, we remark that shrinking a 2-edge in a α' -edge-critical hypergraph H will yield a 1-edge, although we require that the original hypergraph H contains no 1-edge. For example, let H be a hypergraph with $V(H) = \{v_1, v_2, v_3, v_4\}$ and $E(H) = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_1, v_4\}$, $e_3 = \{v_2, v_4\}$ and $e_4 = \{v_3, v_4\}$. It is easy to see that H is 1-edge-critical. From this point of view, Gallai [9] showed that the complete graph K_{2k+1} on $2k + 1$ vertices is the unique k -edge-critical connected graph G . In particular, K_3 is the only 1-edge-critical graph.

Observed by Erdős and Lovász [8], every intersecting hypergraph is 3-colorable. We can get a 3-coloring by coloring the vertices in an arbitrary edge of a intersecting hypergraph with two colors and then coloring all vertices not in this edge with a third color.

For an integer $r \geq 2$, let \mathcal{H}_r denote the class of all intersecting hypergraphs of rank r with no edge consisting of one vertex. The following five subfamilies of hypergraphs in \mathcal{H}_r are defined in [18] (also see [13]).

$$\begin{aligned}\mathcal{H}^1(r) &= \{H \in \mathcal{H}_r \mid H \text{ is 3-chromatic and } r\text{-uniform}\}. \\ \mathcal{H}^2(r) &= \{H \in \mathcal{H}_r \mid H \text{ is maximal 1-special}\}. \\ \mathcal{H}^3(r) &= \{H \in \mathcal{H}_r \mid H \text{ is 1-special}\}. \\ \mathcal{H}^4(r) &= \{H \in \mathcal{H}_r \mid H \text{ is 1-edge-critical}\}. \\ \mathcal{H}^5(r) &= \{H \in \mathcal{H}_r \mid H \text{ is 1-vertex-critical}\}.\end{aligned}$$

For $r \geq 2$ and for $i \in \{1, 2, 3, 4, 5\}$, the maximum order of a hypergraph in the class \mathcal{H}_r^i denoted by $n^i(r)$. Thus,

$$n^i(r) = \max\{|V(H)| : H \in \mathcal{H}_r^i\}.$$

A result gave by Henning and Yeo [18] is that the five subclasses of hypergraphs in \mathcal{H}_r defined in Definition 4 are nested families, which was also stated in [13] without proof..

Theorem 1.1. ([13, 18]) For $r \geq 2$, $\mathcal{H}_r^1 \subset \mathcal{H}_r^2 \subset \mathcal{H}_r^3 \subset \mathcal{H}_r^4 \subset \mathcal{H}_r^5$.

As an immediate consequence of Theorem 1.1, we have the following inequality chain.

Theorem 1.2. ([18]) For $r \geq 2$, $n^1(r) \leq n^2(r) \leq n^3(r) \leq n^4(r) \leq n^5(r)$.

For $r = 2, 3$, Henning and Yeo [18] proved that the above inequality chain is an equality chain.

Theorem 1.3. ([18]) For $i \in \{1, 2, 3, 4, 5\}$, $n_i(2) = 3$ and $n_i(3) = 7$.

An open problem posed by Henning and Yeo [18] is whether the equalities $n^1(r) = n^2(r) = n^3(r)$ and $n^4(r) = n^5(r)$ for $r \geq 4$ hold. Further, is it true that the inequality chain in Theorem 1.2 is an equality chain for $r \geq 4$?

In this paper we show that $n^2(r) = n^3(r)$ and $n^4(r) = n^5(r)$ for all $r \geq 2$. We also prove a strengthening of the equality $n^4(r) = n^5(r)$ for intersecting r -uniform hypergraphs.

2 1-Special and maximal 1-special hypergraphs

In this section we shall prove that $n^2(r) = n^3(r)$ for all $r \geq 2$. To do this, we first observe the following relationship between the 1-special hypergraphs and maximal 1-special hypergraphs.

Proposition 2.1. A hypergraph H is maximal 1-special if and only if H is 1-special and every minimum transversal in H is an edge of H .

Proof. Let H be a maximal 1-special hypergraph. By Definition 1.1, H is 1-special, and so H is r -uniform. Suppose that there exists a minimum transversal T of H such that T is not an edge of H . Then $|T| = r$. Hence T can be regarded as a missing r -edge of H . Let H' be the hypergraph obtained from H by adding the new r -edge T to H . Clearly, $\alpha'(H') = \alpha'(H) = 1$, contradicting the assumption that H is maximal 1-special.

Conversely, suppose that H is a 1-special hypergraph and every minimum transversal in H is an edge of H . We show that H is maximal 1-special. If not, then there exists a missing r -edge of H and adding e to H does not increase matching number. This implies that e has a nonempty intersection with every edge of H . Hence e is a minimum transversal in the resulting hypergraph. This contradicts the assumption that every minimum transversal in H is an edge of H . \square

By Proposition 2.1, we now show that $n^2(r) = n^3(r)$ for all $r \geq 2$.

Theorem 2.1. $n^2(r) = n^3(r)$ for all $r \geq 2$.

Proof. By Theorem 1.2, we have $n^2(r) \leq n^3(r)$ for all $r \geq 2$. To establish the opposite inequality, we show that for an arbitrary 1-special hypergraph H , there exists a maximal 1-special hypergraph H^* such that $V(H) = V(H^*)$.

Let H be an arbitrary 1-special hypergraph, i.e., $H \in \mathcal{H}_r^3$. By Definition 1.1, we have $\tau(H) = r$. If H is maximal 1-special, there is nothing to prove. Otherwise, by Proposition 2.1, there must exist a minimum transversal, say T , of H such that T is not an edge of H . We add T to H as a new edge of H , and denote by H' the resulting hypergraph. Clearly, $\tau(H') = \tau(H) = r$, so H' is still 1-special. If H' is maximal 1-special, then let $H^* = H'$, we are done. Otherwise, by Proposition 2.1 again, there exists a minimum transversal T' of H' such that T' is not an edge of H' . As above, we obtain a 1-special hypergraph H'' by adding T' to H' as a new edge of H' . We repeat the procedure until no minimum transversal T^* in the resulting hypergraph H^* such that T^* is not an edge of H^* . Then H^* is a maximal 1-special hypergraph such that $V(H) = V(H^*)$, as desired. Therefore, we have $n^2(r) \geq n^3(r)$ for all $r \geq 2$. \square

3 1-Edge-critical and 1-vertex-critical hypergraphs

In this section we shall show that $n^4(r) = n^5(r)$ for all $r \geq 2$.

By Definition 1.2, we immediately have the following observation.

Observation 3.1. *A hypergraph H is 1-edge-critical if and only if for any $e \in E(H)$ and any $v \in e$, there exists an edge $f \in E(H)$ such that $e \cap f = \{v\}$.*

Henning and Yeo [18] provided a necessary and sufficient condition for a hypergraph to be 1-vertex-critical.

Lemma 3.1. ([18]) *For all $r \geq 2$, a hypergraph $H \in H_r^5$ if and only if $H \in H_r$ and $qd(v) \geq 2$ for all $v \in V(H)$.*

Lemma 3.2. *For every 1-vertex-critical hypergraph $H \in H^5(r)$, there exists a 1-edge-critical hypergraph $H' \in H^4(r')$ such that $r' \leq r$ and $V(H) = V(H')$.*

Proof. There is nothing to prove if H is 1-edge-critical hypergraph, so we may assume that H is not 1-edge-critical. By Definition 1.2, there exists an edge e of H and a vertex $u \in e$ such that u -shrinking e does not increase the matching number. So every edge f that contains the vertex u satisfies $|e \cap f| \geq 2$ by Observation 3.1. Since $H \in H^5(r)$, we have $qd(u) \geq 2$ by Lemma 3.1. This implies that the edge e contribute zero to the quasidegree of u , so u -shrinking e does not decrease the quasidegree of each vertex in H . Now we replace e by the new edge e_1 obtained from e by u -shrinking e . Let $H_1 = (H - e) \cup \{e_1\}$. Then $V(H) = V(H_1)$ and H_1 is still 1-vertex-critical, but H_1 possibly has smaller rank than H . If H_1 is not 1-edge-critical, then, by repeating this process of shrinking edges, we obtain a 1-edge-critical hypergraph H' with $V(H) = V(H')$ and rank r' , where $r' \leq r$, as desired. \square

The following lemma is the key to the proof of the main result in this section.

Lemma 3.3. $n^4(r+1) > n^4(r)$ for all $r \geq 2$.

Proof. Let H be a hypergraph in \mathcal{H}_r^4 with $|V(H)| = n^4(r)$. By the definition of 1-edge-critical hypergraphs, we have $d(v) \geq 2$ for each vertex $v \in H$. Let e be an arbitrary r -edge in H . We distinguish the following cases depending on the degree of vertices in e .

Case 1: $d(u) = 2$ for all $u \in e$.

In this case, since H is an intersecting hypergraph and each vertex of e has degree two, H contains exactly $r+1$ edges. Let $E(H) = \{e_0, e_1, \dots, e_r\}$ where $e_0 = e$. We construct a 1-edge-critical hypergraph with rank $r+1$ as follows.

Let H' be the hypergraph obtained from H by adding $r+1$ new vertices x_0, x_1, \dots, x_r to H and a new edge $\{x_0, x_1, \dots, x_r\}$ to H and, replacing e_i by $e_i \cup \{x_i\}$ for $i = 0, 1, \dots, r$. By our construction, clearly H' is intersecting and it has rank $r+1$. For notational convenience, let $e'_{r+2} = \{x_0, x_1, \dots, x_r\}$, $e'_i = e_i \cup \{x_i\}$ for all i , $0 \leq i \leq r$. Then $E(H') = \{e'_0, e'_1, \dots, e'_{r+2}\}$. We claim that H' is 1-edge-critical hypergraph of rank $r+1$.

By the construction again, we see that $d(x_i) = qd(x_i) = 2$ for all i , $0 \leq i \leq r$, so x_i -shrinking any edge e'_i containing x_i increases the matching number. Therefore, it remains to show that for an arbitrary $e'_i \in E(H')$ ($i \neq r+2$) and $u \in e$ ($u \neq x_i, 0 \leq i \leq r$), u -shrinking the edge e'_i

increases the matching number. This is indeed so. In fact, since H is 1-edge-critical, u -shrinking the edge e_i will increase the matching number. Let $\{e_i \setminus \{u\}, e_j\}$ be a matching in the resulting hypergraph when we u -shrink the edge e_i . Then clearly $\{e'_i \setminus \{u\}, e'_j\}$ is a matching in the resulting hypergraph when we u -shrink the edge e'_i . Hence $H' \in H^4(r+1)$. Consequently, $n_{r+1}^4 \geq |V(H')| = |V(H)| + r + 1 = n^4(r) + r + 1 > n^4(r)$.

Case 2: There exists a vertex $u \in e$ such that $d(u) \geq 3$.

Since $H \in H^4(r)$, u -shrinking the edge e will increase the matching number, so there exists an edge f such that $f \cap e = \{u\}$. Let $\{e_0, e_1, \dots, e_{d(u)-1}\}$ be the set of all edges containing u , where e, f are renamed as $e_0, e_{d(u)-1}$, respectively. We first construct an intersecting hypergraph H' as follows. Let H' be the hypergraph obtained from H by adding two new vertices $\{x, y\}$ and replacing e by $e_0 \cup \{x\}$, $e_{d(u)-1}$ by $e_{d(u)-1} \cup \{y\}$, e_i by $(e_i \setminus \{u\}) \cup \{x, y\}$ for $i = 1, 2, \dots, d(u) - 2$. By the above construction, clearly H' is intersecting and has rank $r + 1$.

For notational convenience, we write $e'_0, e'_{d(u)-1}$ and e'_i for $e_0 \cup \{x\}$, $e_{d(u)-1} \cup \{y\}$ and $(e_i \setminus \{u\}) \cup \{x, y\}$ ($1 \leq i \leq d(u) - 2$), respectively. Let $S = \{e'_0, e'_1, \dots, e'_{d(u)-1}\}$. Then $E(H') = (E(H) \setminus \{e_0, e_1, \dots, e_{d(u)-1}\}) \cup S$.

To obtain the desired 1-edge-critical hypergraph. We refine the edges in S by the following procedure.

Step 1. First check every edge e'_i ($0 \leq i \leq d(u) - 1$) of S one by one in order. For $e'_i \in S$, if x -shrinking e'_i does not increase the matching number, namely $|e'_i \cap e'_j| \geq 2$ for each edge containing x by Observation 3.1, then we replace e'_i by $e'_i \setminus \{x\}$ and rename $e'_i \setminus \{x\}$ as e'_i .

Let S' (possibly empty) denote the set of the remaining edges e'_i by the above procedure. Then clearly every edge e'_i in S' other than $e'_{d(u)-1}$ still contains the vertices x, y .

Step 2. Further check every edge e'_i of S' one by one in order. If y -shrinking e'_i does not increase the matching number, namely $|e'_i \cap e'_j| \geq 2$ for each edge containing y , then we replace e'_i by $e'_i \setminus \{y\}$ and rename $e'_i \setminus \{y\}$ as e'_i .

When the procedure terminates, we denote the final resulting hypergraph by H'' . By the construction, either both or one of $\{x, y\}$ is still in H'' . Furthermore, it is easy to see that possibly $qd(x) = 0$ (when e_i and e_0 are overlapping for all $1 \leq i \leq d(u) - 1$) or $qd(y) = 0$ in H'' . Specifically, if $qd(x) = 0$, then $qd(y) \geq 2$ in H'' ; if $qd(y) = 0$, then $qd(x) \geq 2$ in H'' . We show that H'' is a 1-edge-critical hypergraph with rank $r + 1$.

We first show that H'' is intersecting. It suffices to show that every two distinct edges of $\{e'_0, e'_1, \dots, e'_{d(u)-1}\}$ in H'' have a non-empty intersection since H is intersecting. By the construction of H'' , $u \in e'_0 \cap e'_{d(u)-1}$. We consider every edge e'_i , for $1 \leq i \leq d(u) - 2$. If $e'_i \cap \{x, y\} = \{y\}$, then e'_i and each vertex e'_j ($1 \leq j \leq d(u) - 2$) containing y share at least the vertex y in common and, e'_i and each vertex e'_j ($0 \leq j \leq d(u) - 2$) containing x (possibly containing y) share the common vertex y or intersect within e'_j . If $e'_i \cap \{x, y\} = \{x\}$, then e'_i

and each edge e'_j ($0 \leq j \leq d(u) - 1$) containing x share at least the vertex x in common and, e'_i and each vertex e'_j ($0 \leq j \leq d(u) - 2$) containing y (possibly containing x) share the common vertex x or intersect within e'_j . If $e'_i \cap \{x, y\} = \{x, y\}$, then e'_i and e'_j intersect in x or y . Thus H'' is intersecting.

In order to show that H'' is 1-edge-critical, since H is 1-edge-critical, it is enough to show that for any edge e'_i ($0 \leq i \leq d(u) - 1$) containing both or one of $\{x, y\}$ and any vertex $v \in e'_i \cap \{x, y\}$ in H'' , v -shrinking e'_i can increase the matching number. Indeed, if $x \in e'_i$, then x -shrinking e'_i can increase the matching number, for otherwise we would delete the vertex x from e'_i by Step 1. If $\{x, y\} \subset e'_i$ or $y \in e'_{d(u)-1}$, then y -shrinking e'_i can increase the matching number, for otherwise we would delete the vertex y from e'_i by Step 2. If $e'_i \cap \{x, y\} = \{y\}$ ($i \neq d(u) - 1$), then there exists an edge e'_j containing y such that $e'_i \cap e'_j = \{y\}$, namely y -shrinking e'_i can increase the matching number. Hence, H'' is 1-edge-critical.

Note that H'' contains at least one vertex in $\{x, y\}$, so $|V(H'')| \geq |V(H)| + 1$. Since H is a hypergraph with maximum order in \mathcal{H}_r^4 . This implies that H'' has rank $r + 1$. Therefore, $n^4(r + 1) \geq |V(H'')| > |V(H)| = n^4(r)$. \square

Theorem 3.1. *For any integers $r \geq 2$, $n^4(r) = n^5(r)$.*

Proof. By Theorem 1.2, we have $n^4(r) \leq n^5(r)$. Let $\mathcal{H} \in H^5(r)$ with $|V(H)| = n^5(r)$. By Lemma 3.2, there exists a 1-edge-critical hypergraph $H' \in H^4(r')$ such that $r' \leq r$ and $V(H) = V(H')$, so $|V(H')| = n^5(r)$. By Lemma 3.3, we have $n^4(r) \geq n^4(r') \geq |V(H')| = n^5(r)$. Therefore, $n^4(r) = n^5(r)$. \square

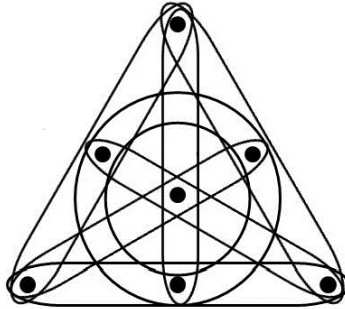


Figure 1: Fano plane: A non-minimal 1-vertex-critical hypergraph

Now we give a strengthening of Theorem 3.1. We shall show that the maximum order $n^5(r)$ of 1-vertex-critical hypergraphs is the same as that of 1-vertex-critical r -uniform hypergraphs.

We call a 1-vertex-critical hypergraph H is *minimal* if H contains no edge e such that $H - e$ is 1-vertex-critical. For $r \geq 2$, let $\mathcal{H}_{uni}^5(r)$ and $\mathcal{H}_{min}^5(r)$ be the sets of 1-vertex-critical r -uniform and minimal 1-vertex-critical hypergraphs in $\mathcal{H}^5(r)$, respectively. Obviously, $\mathcal{H}_{uni}^5(r) \subseteq \mathcal{H}^5(r)$, $\mathcal{H}_{min}^5(r) \subseteq \mathcal{H}^5(r)$, but $\mathcal{H}_{uni}^5(r) \not\subseteq \mathcal{H}_{min}^5(r)$. Fano plane in Fig. 1 clearly is a non-minimal 1-vertex-critical hypergraph. Let $n_r^5 = \max\{|V(H)| : H \in \mathcal{H}_{uni}^5(r)\}$ and $n_r^4 = \max\{|V(H)| : H \in \mathcal{H}^4(r) \text{ and } H \text{ is } r\text{-uniform}\}$.

Lemma 3.4. *For every $H \in \mathcal{H}^5(r)$, there exists an $H' \in \mathcal{H}_{min}^5(r)$ such that $V(H) = V(H')$, and every edge $e \in E(H')$ contains at least a vertex v such that $qd_{H'-e}(v) = 0$.*

Proof. If H is minimal 1-vertex-critical, we are done. Otherwise, there exists an edge $e_1 \in E(H)$ such that $qd_{H-e_1}(v) \geq 2$ for all $v \in V(H - e_1)$ in $H - e_1$. We delete the edge e_1 from H . By Lemma 3.1, the hypergraph $H - e_1$ is still 1-vertex-critical and $V(H) = V(H - e_1)$. If $H - e_1$ still contains an edge e_2 such that $qd_{H-\{e_1, e_2\}}(v) \geq 2$ for all $v \in V(H - \{e_1, e_2\})$, then we delete the edge e_2 from $H - e_1$. Then, by Lemma 3.1, $H - \{e_1, e_2\}$ is still 1-vertex-critical. By repeating this above process of deleting edges in H , we obtain a hypergraph H' still 1-vertex-critical but H' contains no edge e such that $H' - e$ is 1-vertex-critical. Therefore, $H' \in \mathcal{H}_{min}^5(r)$ and $V(H) = V(H')$. \square

By Lemma 3.4, it is easy to see that $n^5(r) = \max\{|V(H)| : H \in \mathcal{H}_{min}^5(r)\}$. Furthermore, we have the following lemma.

Lemma 3.5. *For $r \geq 2$, every hypergraph H in $\mathcal{H}_{min}^5(r)$ with $|V(H)| = n^5(r)$ is r -uniform.*

Proof. Suppose, to the contrary, that H contains a t -edge e such that $t < r$. Since H is minimal 1-vertex-critical, there exists a vertex u in e such that $qd_{H-e}(u) < 2$ by Lemma 3.4, hence there exists an edge f of H whose intersecting with e is only u , i.e., $e \cap f = \{u\}$. Clearly, $f \setminus \{u\}$ is a transversal in $H - e$. Note that H has rank r , without loss of generality, we may assume that f is an r -edge of H .

Let H' be the hypergraph obtained from H by adding a new vertex v and a new edge $(f \setminus \{u\}) \cup \{v\}$ to H and replacing e by $e \cup \{v\}$. By the construction of H' , $V(H') = V(H) \cup \{v\}$ and $E(H') = (E(H) \setminus \{e\}) \cup \{e \cup \{v\}, (f \setminus \{u\}) \cup \{v\}\}$. Hence $|V(H')| = n^5(r) + 1$ and H' has rank r . Note that $f \setminus \{u\}$ is a transversal of $H - e$, so $(f \setminus \{u\}) \cup \{v\}$ meets with all edges of H' . Hence H' is a intersecting hypergraph with rank r . On the other hand, we note that $d(v) = qd(v) = 2$ and $qd(u) \geq 2$ for all $u \in H'$. Thus H' is still 1-vertex-critical, i.e., $H' \in \mathcal{H}^5(r)$. But then $|V(H')| \leq n^5(r)$, which is a contradiction. \square

By Lemma 3.5 and Theorem 3.1, we obtain a strengthening of Theorem 3.1.

Theorem 3.2. *For any integers $r \geq 2$, $n_r^4 = n^4(r) = n^5(r) = n_r^5$.*

Proof. By Lemma 3.5, we have $n_r^5 = n^5(r)$. Let $H \in \mathcal{H}_{min}^5(r)$ is a hypergraph with order n_r^5 . We claim that H is 1-edge-critical. If not, then there exists an edge e and a vertex $v \in e$ such that replacing e by e' which obtained from v -shrinking e dose not increase the matching number. It implies that e contribute zero to the quasidegree of v . Hence, there exists a hypergraph $H' \in \mathcal{H}^5(r)$ with $|V(H')| = n_r^5$ such that H' there exists an $(r-1)$ -edge e' . But this contradicts Lemma 3.5. Thus $n^5(r) = n_r^5 \leq n_r^4 \leq n^4(r)$. By Theorem 3.1, $n_r^4 = n^4(r) = n^5(r) = n_r^5$. \square

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